APPROXIMATE SOLUTION OF AN INTERNAL PROBLEM IN LAMINAR BOUNDARY LAYER THEORY

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Laminar flow of a liquid in a channel of arbitrary shape is examined with account for interaction of the boundary layer with the core. A simple approximate method is described for calculating the boundary layer in a channel with arbitrary generator.

Integral methods permit a quite simple calculation of the boundary layer in the case of the external problem (flow over a body). In principle, these methods may also be applied to the internal problem (flow in channels). In this case, however, difficulties arise from the need to calculate the interaction between the boundary layer and the external flow. The interaction is usually calculated by the method of successive approximations [1, 2]. Several iterations may be necessary in a number of problems, in order to obtain the requisite accuracy, or it may even be necessary to apply the method not to the channel as a whole, but to its separate parts. This applies particularly to the design of diffusors [2].

In the present paper the problem of flow of a liquid in a channel is solved with allowance for interaction by reducing the characteristic system of equations to a single integro-differential equation. For channels with straight generators the equation obtained has been solved numerically on an electronic computer. Comparison of the results of the calculation with data obtained without allowance for interaction permits an assessment of the back effect of the boundary layer on the external flow. Relations have been established for the point of separation and the point at which boundary layers on opposite sides merge. We have constructed a simple approximate method of calculating the boundary layer with allowance for interaction in a plane or axisymmetric channel of arbitrary shape. The problem for a turbulent boundary layer may be solved similarly.

Let us examine the flow in a plane symmetric channel of arbitrary shape in the case when the boundary layers do not meet at the entrance. In writing the boundary layer equations, it is usual to take the longitudinal coordinate along the wall, and the transverse one in a direction perpendicular to it (coordinates x', y'). For the internal problem it is more convenient, however, to have a coordinate system x, y in which the x axis coincides with the axis of symmetry. It is easily verified that at the wall slope angles usually encountered, the equations written in the two systems practically coincide, and the pressure change along the y axis may be neglected, if we consider, as is usual, that the pressure along the y' axis in the boundary layer is constant ($\partial p/\partial y' = 0$). In fact, considering, for simplicity, a channel with a straight wall (Fig. 1) inclined at an angle $\Theta/2$ to the axis, we have

$$\frac{\partial p}{\partial y'} = \frac{\partial p}{\partial x} \sin\frac{\Theta}{2} + \frac{\partial p}{\partial y} \cos\frac{\Theta}{2}.$$

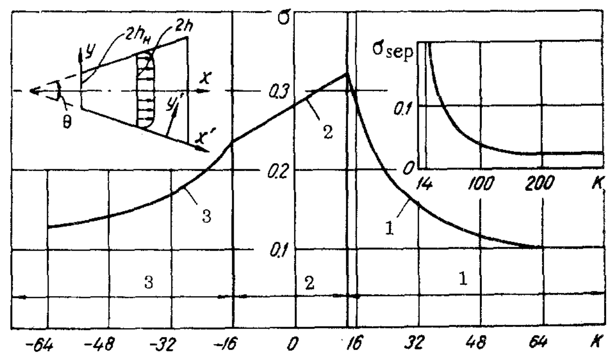

Fig. 1. Dependence of the dimensionless coordinates of the separation point σ_{sep} and of the meeting point of the boundary layers $\sigma = \sqrt{x/h_H}\,\mathrm{Re}$ on the number $K = \mathrm{Re}\,\mathrm{tg}\,(\Theta/2)$: 1) separation region; 2) and 3) regions where the boundary layers have and have not merged.

If we further assume that

$$\frac{\partial p}{\partial x} \approx (\rho U^2/h)\,\mathrm{tg}\,\frac{\Theta}{2},$$

then for the pressure change in the y direction, with boundary layer thickness δ, we obtain the following estimate:

$$\frac{\Delta p}{\rho U^2} \approx \frac{\delta}{h}\,\mathrm{tg}^2\,\frac{\Theta}{2}.$$

Even when $\delta/h = 1$ and the diffusor angle $\Theta = 20°$, this quantity is equal to 3%. It may similarly be verified that the momentum equation written in x, y coordinates coincides with its usual form correct to $\cos(\Theta/2)$.

Then to calculate the boundary layer we have the following system of equations:

$$\frac{d\delta^{**}}{dx} + \frac{U'}{U}(2\delta^{**} + \delta^*) = \frac{\tau_w}{\rho U^2}, \qquad (1)$$

$$Q = U(h - \delta^*). \qquad (2)$$

If some one-parameter method is used, the momentum equation (1) can be linearized and its solution represented in the form [3]

$$\delta^{**2} = \delta_H^{**2}\left(\frac{U_H}{U}\right)^b + \frac{a\nu}{U^b}\int_0^x U^{b-1}dx, \qquad (3)$$

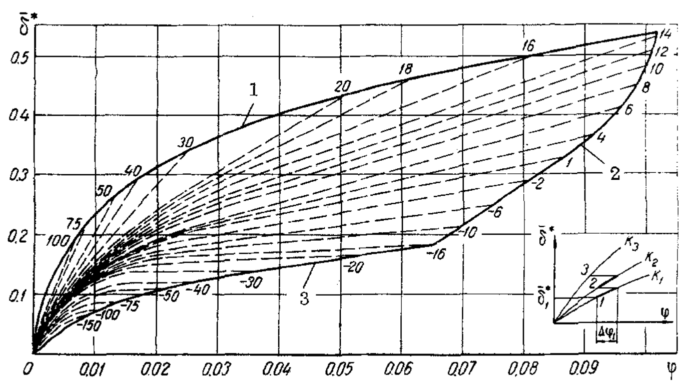

Fig. 2. Dependence of $\bar{\delta}^*$ on $\varphi = x/h_H$ Re and K: 1) separation; 2) meeting; 3) no meeting; broken lines—values of K.

where a and b are constants, and subscript H refers to the entrance section of the channel. The system of Eqs. (2) and (3) contains three unknowns: U, δ^*, and δ^{**}. It is necessary to use the dependence of the quantity $H = \delta^*/\delta^{**}$ on some shape factor as the missing equation. The most convenient shape factor proved to be that determined by the displacement thickness δ^*,

$$g = U'\delta^{*2}/\nu. \tag{4}$$

Calculations carried out for four different sets of boundary layer velocity profiles (profiles obtained from exact solutions for linear and power-law velocity distributions in the external flow, and profiles represented by a polynomial of fourth degree and by the polynomial proposed in [4]) show that the dependence of $H^2(g)$ may be approximated with sufficient accuracy by the straight line

$$H^2 = c_1 - c_2 g \quad (c_1 = 0.149;\ c_2 = 0.07). \tag{5}$$

Using equalities (2), (4), and (5), and introducing the new variables

$$z = Q/h_H U = \bar{h}(1 - \bar{\delta}^*),\ \ \varphi = \bar{x}/\mathrm{Re}, \tag{6}$$

we obtain, after transformation, in place of (3) the following integro-differential equation:

$$c_1 z^2 (\bar{h} - z)^2 - c_2 \frac{dz}{d\varphi}(\bar{h} - z)^4 - \frac{\bar{\delta}_H^{**2}}{(1 - \bar{\delta}_H^*)^b} z^{b+2} - a z^{b+2}\int_0^{\varphi} z^{1-b} d\varphi = 0, \tag{7}$$

which must be solved under the boundary condition

$$\varphi = 0, \quad z = 1 - \bar{\delta}_H^*.$$

For a given channel shape $\bar{h} = \bar{h}(\bar{x})$ and entrance conditions ($\bar{\delta}_H$; $\bar{\delta}_H^{**}$), Eq. (7) determines the quantity z, and therefore $\bar{\delta}^*$.

Values of the constants a and b in (7) are determined by the selected one-parameter method of calculating the boundary layer. The choice of method, generally speaking, cannot be very important, since it is known [3] that different one-parameter methods give quite close results, except in the region close to separation. Nevertheless, to increase the reliability of the calculations in this region too, preference should be given to a method based on the set of profiles obtained for some particular problem close to that considered. For this reason, we shall use the method which employs the set of profiles in a flow with a linear velocity law in the external stream. This flow may be regarded as a flow in some channel [5]. Accordingly, it has been assumed [6] that $a = 0.44$; $b = 6$.

Let us examine the case when the channel generators are straight lines and form an angle Θ. Then

$$\bar{h} = 1 + K\varphi,$$

where

$$K = \mathrm{Re}\,\mathrm{tg}\,\frac{\Theta}{2}, \tag{8}$$

and $K > 0$ corresponds to an expanding, $K < 0$ to a contracting channel. If the profile at the channel entrance is uniform ($\bar{\delta}_H^* = \bar{\delta}_H^{**} = 0$), the third term in (7) vanishes, and (7) assumes the form

$$0.149 z^2 (1 + K\varphi - z)^2 - 0.07\frac{dz}{d\varphi}(1 + K\varphi - z)^4 - 0.44 z^8 \int_0^{\varphi}\frac{d\varphi}{z^5} = 0. \tag{9}$$

In this simplest case z proves to depend only on the single parameter K and may conveniently be tabulated. Equation (9) was integrated numerically on a computer.* To perform the calculations it was convenient to go over to the new variable

$$\sigma = \sqrt{\varphi} = \sqrt{x/h_H \mathrm{Re}} \tag{10}$$

*The calculations were made by my junior scientific associate, O. D. Lipovetska.

and to replace (9) by the system

$$z' = \frac{0.1693(1+K\sigma^2 - z)^2 - \sigma z^2 \omega}{0.03976(1+K\sigma^2 - z)^4}, \quad \omega' = \frac{\sigma}{z^5}, \tag{11}$$

where the primes denote differentiation with respect to σ, and the integral appearing in (9) is denoted by ω. The boundary condition then takes the form $\sigma = 0$, $z = 1$, $\omega = 0$. At the point $\sigma = 0$ the first equality in (11) does not permit determination of the value of z, and therefore a numerical calculation is possible only from a certain $\sigma_1 > 0$ on. Near $\sigma = 0$ the solution of (9) may be represented by the series

$$\begin{aligned} z = {} & 1 - 1.7185\sigma + (4.962 + K)\sigma^2 - \\ & -(28.327 + 4.199K)\sigma^3 + (246.86 + 37.785K)\sigma^4 - \\ & -(3279.6 + 326.93K + 8.432K^2)\sigma^5 + \ldots \end{aligned} \tag{12}$$

The calculation was performed either up to the separation point, or to the meeting point of the boundary layers. Then the separation point was determined by the separation value of the shape factor (4), $g = 1.232$, and the meeting point of the boundary layers by the dimensionless value of the boundary layer thickness $\delta/h = 1$. To find the latter we used the relation

$$\delta^*/\delta = 0.34 + 0.16g,$$

found with the help of the set of profiles corresponding to a power-law velocity distribution in the external flow.

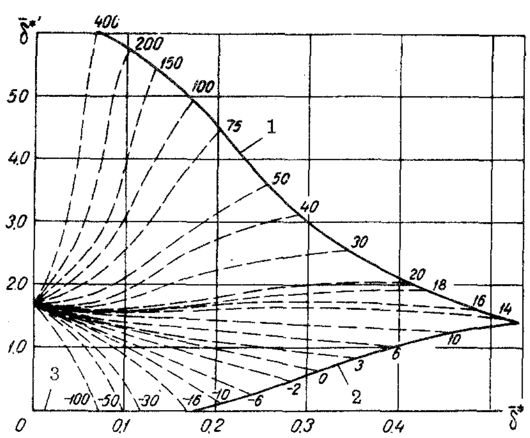

Fig. 3. Graph for determining $\bar\delta^{*\prime}$ (1–3—see Fig. 2; broken lines—values of K).

It can be seen from Fig. 1 that the boundary layers meet when K < 14. Calculation shows that in channels with K < −16, the boundary layers merge close to the point where the walls of the convergent channel converge, so that in practice it may be assumed that in these channels the layers do not meet. When K > 14, separation of the boundary layer occurs before the layers have come together. All three regions are indicated in Fig. 1, the curve in the region K < −16 giving the dependence of the coordinate of the point at which the walls of the convergent channel come together on K. Approximating to the curves given in Fig. 1, we obtain the following formulas for the coordinates of the meeting and separation points:

$$\frac{l}{l_0} = \left(1 + 0.01\mathrm{Re}\,\mathrm{tg}\,\frac{\Theta}{2}\right)^2 \left(-16 < \mathrm{Re}\,\mathrm{tg}\,\frac{\Theta}{2} < 14\right), \tag{13}$$

$$\frac{x_{sep}}{h_H} = 3\mathrm{Re} \Big/ \left(\mathrm{Re}\,\mathrm{tg}\,\frac{\Theta}{2} - 4\right)^{10.7} \left(\mathrm{Re}\,\mathrm{tg}\,\frac{\Theta}{2} > 14\right), \tag{14}$$

where l_0 is the coordinate of the meeting point for a plane tube.

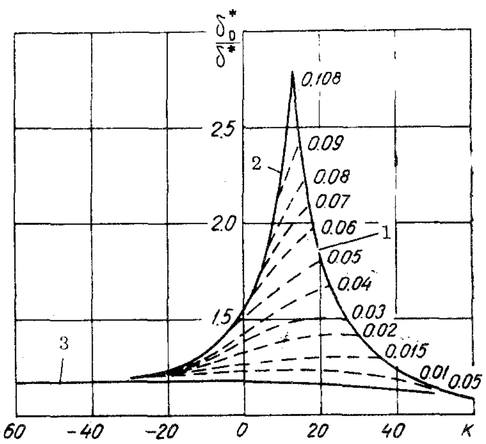

Fig. 4. Back action of boundary layer in various channels (1–3—see Fig. 2; broken lines—values of φ).

Figures 2 and 3 show the results of calculations of the displacement thickness $\bar\delta^* = \delta^*/h$ and the derivative $\bar\delta^{*\prime} = d\bar\delta^*/d\sigma$. Using these data, it is not difficult to determine all the required properties of the flow in the channel. The dimensionless static pressure $\bar p = p - p_H \Big/ \frac{\rho U_H^2}{2}$ at any section of the channel may be computed from the formula

$$\bar p = 1 - 1/\bar h^2 (1 - \bar\delta^*)^2 .$$

The value of the dimensionless pressure $\bar p$ at the end of the channel determines the recovery pressure coefficient η, and therefore the value of the total loss coefficient (including outlet velocity losses)

$$\xi_n = 1 - \eta.$$

The loss coefficient inside the channel may be calculated from the formula [1]

$$\xi = \bar\delta^{***}/\bar h^2 (1 - \bar\delta^*)^3. \tag{15}$$

The value of $\bar\delta^{***}$ in this equality may be found with the aid of the relation [7]

$$\bar\delta^{***}/\bar\delta^* = 0.6 - 0.17g.$$

The shape factor g is then given by the equality

$$g = \frac{\bar\delta^{*2}}{(1 - \bar\delta^*)^2} \left[K(1 - \bar\delta^*) - \bar h\,\frac{\bar\delta^{*\prime}}{2\sigma}\right]. \tag{16}$$

The calculated dimensionless static pressure p_{sep} at the separation point is as follows:

$K = \mathrm{Re}\,\mathrm{tg}\,\frac{\Theta}{2}$	14	20	30	40
$\bar{p}_{sep}$	0.29	0.29	0.29	0.30

50	75	100	200	300
0.30	0.31	0.30	0.29	0.29

It can be seen from these data that $\bar{p}_{sep}$ is practically constant and does not depend on the Re number, nor on the divergence angle Θ of the diffusor. This fact can evidently be used for an experimental determination of the separation point. It is seen from the table that for the laminar flow case examined

$$\bar{p}_{sep} \approx 0.3.$$

Figure 4 compares calculations made with and without allowance for the back action of the boundary layer. Here δ_0^* denotes the value of δ^* found without allowance for the back action of the boundary layer, under the common assumption that H does not depend on the pressure gradient. It can be seen from Fig. 4 that the back action is very different in the different flows. In certain cases failure to allow for the back action may lead to a value of δ^* that is considerably too high (a factor of three), while in other cases ($K < 0$) the back action proves to be comparatively insignificant. It grows with distance from the entrance section and is greatest for channels with $0 < K < 30$.

Let us now examine the general case when the channel generators are curved. It is then necessary, generally speaking, to integrate (7) numerically. It is possible, however, on the basis of the graphs presented above, to construct a simple approximate method of calculation, if we make the usual assumption on which the one-parameter methods are based. According to this assumption, we assume that over length dx in an arbitrary channel, the boundary layer develops in the same way as on some section of a channel with straight generators, provided that one of the characteristic dimensionless boundary layer thicknesses (e.g., $\bar{\delta}^*$) and the quantity $K = \mathrm{Re}\,\mathrm{tg}\,(\Theta/2)$ have identical values for both sections. This assumption, with the aid of the graph of Fig. 2, allows us to determine approximately the value of $\bar{\delta}^*$ at any section of a channel of arbitrary shape. For this purpose we divide the channel into small segments within which the angle Θ (and thus K) may be considered constant. Knowing the values $\bar{\delta}_1^*$ and K_1 at the beginning of the first segment, we can find the corresponding point 1 on the graph (Fig. 2). By moving a distance $\Delta\varphi_1$ (dimensionless length of the first segment) along the curve K = = const through the point 1, we find the value $\bar{\delta}_2^*$ at the start of the next segment. Moving further along the horizontal to the curve K = const, corresponding to K_2, we find the point 2 corresponding to the start of the second segment. Moving along the curve K = = const a distance $\Delta\varphi_2$, we find the value of $\bar{\delta}_3^*$ at the start of the third segment, and so on. The value of the shape factor at any section of the channel may be determined from the formula, analogous to (16),

$$g = \frac{\bar{\delta}^{*2}}{2\sigma(1-\bar{\delta}^*)^2}\,[\bar{h}'(1-\bar{\delta}^*) - \bar{h}\bar{\delta}^{*\prime}],$$

the value $\bar{\delta}^{*\prime}$ being determined with the aid of Fig. 3 from the known $\bar{\delta}^*$ and K. Knowing $\bar{\delta}^*$ and g, it is easy to determine the separation point, the meeting point of the layers, and all the other necessary quantities.

The results obtained may also be used to calculate axisymmetric channels. If, as is usual, we neglect the influence of transverse channel curvature, the system of equations for this case may be written in the form

$$\frac{d\Delta^{**}}{dx} + \frac{1}{U}\frac{dU}{dx}\Delta^{**}(2+H) = 2R\frac{\tau_w}{\rho U^2}, \qquad (17)$$

$$Q = \frac{\pi U}{2}(R^2 - \Delta^*), \qquad (18)$$

where

$$\Delta^{*} = 2\int_0^R \left(1-\frac{u}{U}\right) r dr;\ \Delta^{**} = 2\int_0^R \frac{u}{U}\left(1-\frac{u}{U}\right) r dr.$$

By introducing variables similar to those of Stepanov [8]

$$x_{pl} = 4\int_0^x R^2(t)\,dt,\ y_{pl} = r^2,\ h = R^2,\ Q_{pl} = 2Q/\pi, \quad (19)$$

the case under examination reduces to the plane one, and the system (17), (18) to (1), (2), while

$$\bar{\delta}^*_{pl} = \Delta^*/R^2,\ \bar{\delta}^{**}_{pl} = \Delta^{**}/R^2,\ \tau_w = \tau_{w\,pl} R.$$

If the channel profile is given in the form $R = R(x)$, then, after computing x_{pl}, h, φ_{pl} and

$$K = \frac{Q_{pl}}{\nu}\,\mathrm{tg}\,\frac{\Theta_{pl}}{2} = \frac{Q}{\pi\nu}\,\frac{1}{R}\,\mathrm{tg}\,\frac{\Theta}{2},$$

we perform the calculation for the equivalent plane channel, by dividing it into segments with constant values of K. The values found for the characteristic dimensionless thicknesses $\bar{\delta}_{pl}$ at any section of the plane channel with coordinate $\bar{x}_{pl}$ are numerically equal to the corresponding values of dimensionless areas Δ/R^2 in the axisymmetric channel at the section with coordinate x, which is related to x_{pl} by the first of equations (19).

NOTATION

x, y, x', y' are the coordinates (Fig. 1); Θ is the angle between channel walls; p is the pressure; ρ is the density; U is the flow velocity in potential core; h is the half-height of channel; $\bar{h} = h/h_H$ is the dimensionless channel height; R, r are the outside and variable radii of channel section; Q is the half-mass flowrate of liquid; δ is the boundary layer thickness; δ^* is the displacement thickness; δ^{**} is the momentum thickness; δ^{***} is the energy thickness; $\bar{\delta}^* = \delta^*/h$, $\bar{\delta}^{**} = \delta^{**}/h$ are the dimensionless displacement and momentum thicknesses; $p = p - p_H/\rho U_H^2$ is the dimensionless static pressure; Δ^* is the displacement area; Δ^{**} is the momentum area; Δ^{***} is the energy area; τ_w is the friction stress at wall; ν is the kinematic viscosity; $Re = Q/\nu$ is the Reynolds number; n is the channel expansion ratio.

REFERENCES

1. A. S. Ginevskii, Promyshlennaya aerodinamika, no. 7, 1956.

2. H. Schlichting and K. Cersten, Zeitschrift für Flugwissenschaften, no. 4, 5, 1961.

3. L. G. Loitsyanskii, The Laminar Boundary Layer [in Russian], Fizmatgiz, 1962.

4. L. G. Loitsyanskii, DAN SSSR, **35**, no. 8, 1942.

5. H. Schlichting, Boundary Layer Theory [Russian translation], IL, 1956.

6. A. Sh. Dorfman, collection: Flow of Liquids and Gases [in Russian], Izd. "Naukova dumka," 1965.

7. Truckenbrodt, Ingenieur-Archive, **20**, no. 4, 1952.

8. E. I. Stepanov, PMM, **11**, no. 1, 1947.

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